SOME GEOMETRIC AND TOPOLOGICAL PROPERTIES OF A NEW SEQUENCE SPACE DEFINED BY DE LA VALLÉE-POUSSIN MEAN

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Abstract. The main purpose of this paper is to introduce a new sequence space by using de la Vallée-Poussin mean and investigate both the modular structure with some geometric properties and some topological properties with respect to the Luxemburg norm.

1. Introduction

In summability theory, de la Vallée-Poussin's mean is first used to define the (V, λ)-summability by Leindler [9]. Malkowsky and Savaş [14] introduced and studied some sequence spaces which arise from the notion of generalized de la Vallée-Poussin mean. Also the (V, λ)-summable sequence spaces have been studied by many authors including [6] and [20].

Recently, there has been a lot of interest in investigating geometric properties of several sequence spaces. Some of the recent work on sequence spaces and their geometrical properties is given in the sequel: Shue [21] first defined the Cesáro sequence spaces with a norm. In [11], it is shown that the Cesáro sequence spaces ces_p $(1 \leq p < \infty)$ have Kadec-Klee and Local Uniform Rotundity(LUR) properties. Cui-Hudzik-Pluciennik [4] showed that Banach-Saks of type p property holds in these spaces. In [15], Mursaleen et al studied some geometric properties of normed Euler sequence space. Karakaya [7] defined a new sequence space involving lacunary sequence space equipped with the Luxemburg norm and studied Kadec-Klee(H), rotund(R) properties of this space. Quite recently, Sanhan and Suantai [19] generalized normed Cesáro sequence spaces to paranormed sequence spaces by making use of Köthe sequence spaces. They also defined and investigated modular structure and some geometrical properties of these generalized sequence spaces. In addition, some related papers on this topic can be found in [1],[2],[5],[16],[17] and [23].

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In this paper, our purpose is to introduce a new sequence space defined by de la Vallée-Poussin's mean and investigate some topological and geometric properties of this space.

The organization of our paper is as follows: In the first section, we introduce some definition and concepts that are used throughout the paper. In the second section, we construct a new paranormed sequence space and investigate some geometrical properties of this space. Finally, in the third section, we construct the modular space $V_\rho(\lambda;p)$ which is obtained by paranormed space $V(\lambda;p)$ and we investigate the Kadec-Klee property of this space. We also show that the modular space $V_\rho(\lambda;p)$ is a Banach space under the Luxemburg norm. Also in this section, we investigate the Banach-Saks of type p property of the space $V_p(\lambda)$.

2. Preliminaries, Background and Notation

The space of all real sequences $x=(x(i))_{i=1}^{\infty}$ is denoted by ℓ^0. Let $(X,\|.\|)$ (for the brevity $X=(X,\|.\|)$) be a normed linear space and let $S(X)$ and $B(X)$ be the unit sphere and unit ball of X, respectively.

A Banach space X which is a subspace of ℓ^0 is said to be a Köthe sequence space, if (see [10]) ;

(i) for any $x\in\ell^0$ and $y\in X$ such that $|x(i)|\leq|y(i)|$ for all $i\in\mathbb{N}$, we have $x\in X$ and $\|x\|\leq\|y\|$,

(ii) there is $x\in X$ with $x(i)>0$ for all $i\in\mathbb{N}$.

We say that $x\in X$ is order continuous if for any sequence (x_n) in X such that $x_n(i)\leq|x(i)|$ for each $i\in\mathbb{N}$ and $x_n(i)\to 0\,(n\to\infty)$, $\|x_n\|\to 0$ holds. A Köthe sequence space X is said to be order continuous if all sequences in X are order continuous. It is easy to see that $x\in X$ is order continuous if and only if $\|(0,0,...,0,x(n+1),x(n+2),...)\|\to 0$ as $n\to\infty$.

A Banach space X is said to have the *Kadec-Klee property* (or property (H)) if every weakly convergent sequence on the unit sphere with the weak limit in the sphere is convergent in norm.

Let $1<p<\infty$. A Banach space is said to have the $Banach-Saks\ type\ p$ or property (BS_p), if every weakly null sequence (x_k) has a subsequence (x_{k_l}) such that for some $C>0$,

$$\left\|\sum_{l=0}^{n}x_{k_l}\right\|<C(n+1)^{\frac{1}{p}}$$

for all $n\in\mathbb{N}$ (see [8]).

For a real vector space X, a functional $\rho:X\to[0,\infty]$ is called a *modular* if it satisfies the following conditions:

i) $\rho(x)=0\Leftrightarrow x=0$,

ii) $\rho(\alpha x)=\rho(x)$ for all $\alpha\in\mathbb{F}$ with $|\alpha|=1$,

iii) $\rho(\alpha x+\beta y)\leq\rho(x)+\rho(y)$ for all $x,y\in X$ and all $\alpha,\beta\geq 0$ with $\alpha+\beta=1$.

Further, the modular ρ is called *convex* if

iv) $\rho(\alpha x+\beta y)\leq \alpha\rho(x)+\beta\rho(y)$ holds for all $x,y\in X$ and all $\alpha,\beta\geq 0$ with $\alpha+\beta=1$.

ρ is a modular in X, we define

$$X_\rho=\left\{x\in X:\ \rho(\lambda x)\to 0 \quad \text{as} \quad \lambda\to 0^+\right\},$$

$$X_\rho^*=\left\{x\in X:\ \rho(\lambda x)<\infty \text{ for some } \quad \lambda>0\right\}.$$

It is clear that $X_\rho\subseteq X_\rho^*$. If ρ is a convex modular, for $x\in X_\rho$, we define

$$||x||_L=\inf\left\{\lambda>0:\rho(\frac{x}{\lambda})\leq 1\right\}$$

and

$$||x||_A=\inf_{\lambda>0}\frac{1}{\lambda}\left(1+\rho(\lambda x)\right).$$

If ρ is a convex modular on X, then $X_\rho=X_\rho^*$ and both $||\cdot||_L$ and $||\cdot||_A$ is a norm on X_ρ for which X_ρ is a Banach space.

The norms $||\cdot||_L$ and $||\cdot||_A$ are called the *Luxemburg norm* and the *Amemiya norm(Orlicz norm)*, respectively.

In addition

$$||x||_L\leq ||x||_A\leq 2||x||_L$$

for all $x\in X_\rho$ holds (see [18]).

A sequence (x_n) of elements of X_ρ is called *modular convergent* to $x\in X_\rho$ if there exists a $\lambda>0$ such that $\rho(\lambda(x_n-x))\to 0$ as $n\to\infty$.

Proposition 2.1. *Let $(x_n)\subset X_\rho$. Then $||x_n||_L\to 0$ (or equivalently $||x||_A\to 0$) if and only if $\rho(\lambda(x_n))\to 0$ as $n\to\infty$, for every $\lambda>0$.*

Proof. See [18, p.15, Th.1]. □

Throughout the paper, the sequence $p=(p_k)$ is a bounded sequence of positive real numbers with $p_k>1$, also $H=\sup_k p_k$ and $M=\max\{1,H\}$.

Besides, we will need the following inequalities in the sequel;

$$|a_k+b_k|^{p_k}\leq K\left(|a_k|^{p_k}+|b_k|^{p_k}\right) \tag{2.1}$$

$$|a_k+b_k|^{t_k}\leq |a_k|^{t_k}+|b_k|^{t_k} \tag{2.2}$$

where $t_k=\frac{p_k}{M}\leq 1$ and $K=\max\{1,2^{H-1}\}$, with $H=\sup_k p_k$.

Now we begin the construction of a new sequence space.

Let $\Lambda=(\lambda_k)$ be a nondecreasing sequence of positive real numbers tending to infinity and let $\lambda_1=1$ and $\lambda_{k+1}\leq\lambda_k+1$.

The generalized de la Vallée-Poussin means of a sequence $x=(x_k)$ are defined as follows:

$$t_k(x)=\frac{1}{\lambda_k}\sum_{j\in I_k}x_k \quad \text{where } I_k=[k-\lambda_k+1,k] \quad \text{for} \quad k=1,2,\dots .$$

We write

$$[V,\lambda]_0 = \left\{ x \in \ell^0 : \lim_{k\to\infty} \frac{1}{\lambda_k} \sum_{j\in I_k} |x_j| = 0 \right\}$$

$$[V,\lambda] = \left\{ x \in \ell^0 : x - le \in [V,\lambda]_0, \text{ for some } l \in \mathbb{C} \right\}$$

and

$$[V,\lambda]_\infty = \left\{ x \in \ell^0 : \sup_k \frac{1}{\lambda_k} \sum_{j\in I_k} |x_j| < \infty \right\}$$

for the sequence spaces that are strongly summable to zero, strongly summable and strongly bounded by the de la Vallée-Poussin method (see [9]). In the special case where $\lambda_k = k$ for $k = 1, 2, ...$ the spaces $[V,\lambda]_0$, $[V,\lambda]$ and $[V,\lambda]_\infty$ reduce to the spaces w_0, w and w_∞ introduced by Maddox [12].

We now define the following new paranormed sequence space:

$$V(\lambda;p) := \left\{ x = (x_j) \in \ell^0 : \sum_{k=1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{j\in I_k} |x_j| \right)^{p_k} < \infty \right\}.$$

The space $V(\lambda;p)$ is reduced to some special sequence spaces corresponding to special cases of sequence (λ_k) and (p_k). For example: If we take $\lambda_k = k$, we obtain the space $ces(p)$ defined by [22]. If we take $\lambda_k = k$ and $p_k = p$ for all $k \in \mathbb{N}$, the space $V(\lambda;p)$ reduces to the space ces_p defined by [21].

3. Some Topological Properties Of The Sequence Space $V(\lambda;p)$

In this section, we will give the topological properties of the space $V(\lambda;p)$. We begin by obtaining the first main result.

Theorem 3.1. *a) The space $V(\lambda;p)$ is a complete paranormed space with paranorm defined by*

$$h(x) := \left(\sum_{k=1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{j\in I_k} |x_j| \right)^{p_k} \right)^{\frac{1}{M}}. \tag{3.1}$$

b) if $p_k = p$; the space $V(\lambda;p)$ reduced to $V_p(\lambda)$ defined by

$$V_p(\lambda) := \left\{ x = (x_j) \in \ell^0 : \sum_{k=1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{j\in I_k} |x_j| \right)^{p} < \infty \right\}.$$

And the space $V_p(\lambda)$ is a complete normed space defined by

$$\|x\|_{V_p(\lambda)} := \left(\sum_{k=1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{j\in I_k} |x_j| \right)^{p} \right)^{\frac{1}{p}} \qquad (1 < p < \infty).$$

Proof. a) The linearity of $V(\lambda;p)$ with respect to coordinatewise addition and scalar multiplication follows from the inequality (2.1). Because, for any $x, y \in V(\lambda;p)$ the following inequalities are satisfied:

$$\left(\sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x_j+y_j|\right)^{p_k}\right)^{\frac{1}{M}} \leq \left(\sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x_j|\right)^{p_k}\right)^{\frac{1}{M}} + \left(\sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|y_j|\right)^{p_k}\right)^{\frac{1}{M}} \tag{2.2}$$

and for any $\alpha \in \mathbb{R}$ (see [13]) we have

$$|\alpha|^{p_k} \leq \max\{1, |\alpha|^M\}. \tag{2.3}$$

It is clear that $h(\theta) = 0$ and $h(x) = h(-x)$ for all $x \in V(\lambda;p)$. Again the inequalities (2.2) and (2.3) yield the subadditivity of h and

$$h(\alpha x) \leq \max\{1, |\alpha|\}h(x).$$

Let (x^m) be any sequence of points of the space $V(\lambda;p)$ such that $h(x^m - x) \to 0$ and (α_n) also be any sequence of scalars such that $\alpha_n \to \alpha$. Then, since the inequality

$$h(x^m) \leq h(x) + h(x^m - x)$$

holds by subadditivity of h, the sequence $(h(x^m))_{m\in\mathbb{N}}$ is bounded and we thus have

$$\begin{aligned} h(\alpha_m x^m - \alpha x) &= \left(\sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|\alpha_m x_j^m - \alpha x_j|\right)^{p_k}\right)^{\frac{1}{M}} \\ &\leq |\alpha_m - \alpha|\, h(x^m) + |\alpha| h(x^m - x). \end{aligned}$$

The last expression tends to zero as $m \to \infty$, that is, the scalar multiplication is continuous. Hence h is paranorm on the space $V(\lambda;p)$.

It remains to prove the completeness of the space $V(\lambda;p)$.

Let (x^n) be any Cauchy sequence in the space $V(\lambda;p)$, where $x = (x_j^n) = (x_1^n, x_2^n, x_3^n, ...)$. Then, for a given $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$ such that

$$h(x^n - x^m) < \frac{\varepsilon}{2}$$

for every $m, n \geq n_0(\varepsilon)$. By using the definition of h, we obtain that

$$\left(\sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x_j^n - x_j^m|\right)^{p_k}\right) < \varepsilon^M$$

for every $m, n \geq n_0(\varepsilon)$. Also we get, for fixed $j \in \mathbb{N}$, $|x_j^n - x_j^m| < \varepsilon$ for every $m, n \geq n_0(\varepsilon)$. Hence it is clear that the sequences (x_j^n) is a Cauchy sequence in $\mathbb{R}$. Since the real numbers set is complete, so we have $x_j^m \to x_j$ for every $n \geq n_0(\varepsilon)$ and as $m \to \infty$. Now we get

$$\left(\sum_{k=1}^{r}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x_j^n - x_j|\right)^{p_k}\right) < \varepsilon^M.$$

If we pass to the limit over the r to infinity and $n \geq n_0(\varepsilon)$ we obtained $h(x^n - x) < \varepsilon$. So, the sequence (x^n) is a Cauchy sequence in the space $V(\lambda; p)$.

It remains to show that the space $V(\lambda; p)$ is complete. Since we have $x = x^n - x^n + x$, we get

$$\sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x_j|\right)^{p_k} \leq \sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x_j^n - x_j|\right)^{p_k} + \sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x_j^n|\right)^{p_k}.$$

Consequently, we obtain $x \in V(\lambda; p)$. This completes the proof.

$b)$ By taking $p_k = p$ in (a), it can be easily shown the proof of (b). □

4. Some Geometric Properties Of The Spaces $V_\rho(\lambda; p)$ And $V_p(\lambda)$.

In this section we construct the modular structure of the space $V(\lambda; p)$ and since the Luxemburg norm is equivalent to usual norm of the space $V_p(\lambda)$, we show that the space $V_p(\lambda)$ has the Banach-Saks type p.

Firstly, we will introduce a generalized modular sequence space $V_\rho(\lambda; p)$ by

$$V_\rho(\lambda; p) := \left\{x \in \ell^0 : \rho(\lambda x) < \infty, \text{ for some } \lambda > 0\right\},$$

where

$$\rho(x) = \left(\sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x_j|\right)^{p_k}\right).$$

It can be seen that $\rho : V_\rho(\lambda; p) \to [0, \infty]$ is a modular on $V_\rho(\lambda; p)$.

Note that the Luxemburg norm on the sequence space $V_\rho(\lambda; p)$ is defined as follows:

$$||x||_L = \inf\left\{\lambda > 0 : \rho(\frac{x}{\lambda}) \leq 1\right\}, \qquad \text{for all } x \in V_\rho(\lambda; p)$$

or equally

$$||x||_L = \inf\left\{\lambda > 0 : \rho(\frac{x}{\lambda}) = \left(\sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x_j|\right)^{p_k}\right) \leq 1\right\}.$$

In the same way we can introduce the Amemiya norm (Orlicz norm) on the sequence space $V_\rho(\lambda; p)$ as follows:

$$||x||_A = \inf_{\lambda > 0}\frac{1}{\lambda}\left(1 + \rho(\lambda x)\right) \qquad \text{for all } x \in V_\rho(\lambda; p).$$

We now give some basic properties of the modular ρ on the space $V_\rho(\lambda; p)$. Also we will investigate some relationships between the modular ρ and the Luxemburg norm on $V_\rho(\lambda; p)$.

Proposition 4.1. *The functional ρ is a convex modular on $V_\rho(\lambda; p)$.*

Proposition 4.2. *For any $x \in V_\rho(\lambda; p)$*

i) if $||x||_L \leq 1$, then $\rho(x) \leq ||x||_L$;

ii) $||x||_L = 1$ if and only if $\rho(x) = 1$.

Proposition 4.3. *For any $x \in V_\rho(\lambda;p)$, we have*
i) If $0 < a < 1$ and $||x||_L > a$, then $\rho(x) > a^H$;
ii) if $a \geq 1$ and $||x||_L < a$, then $\rho(x) < a^H$.

The proofs of the three propositions given above are proved with standard techniques in a similar way as in [19] and [3].

Proposition 4.4. *Let (x_n) be a sequence in $V_\rho(\lambda;p)$. Then:*
i) if $\lim_{n\to\infty} ||x_n||_L = 1$, then $\lim_{n\to\infty} \rho(x_n) = 1$;
ii) if $\lim_{n\to\infty} \rho(x_n) = 0$, then $\lim_{n\to\infty} ||x_n||_L = 0$.

Proof. (i) Suppose that $\lim_{n\to\infty} ||x_n||_L = 1$. Let $\varepsilon \in (0,1)$. Then there exists n_0 such that $1-\varepsilon < ||x_n||_L < 1+\varepsilon$ for all $n \geq n_0$. Since $(1-\varepsilon)^H < ||x_n||_L < (1+\varepsilon)^H$ for all $n \geq n_0$ by the Proposition 4.3 (i) and (ii), we have $\rho(x_n) \geq (1-\varepsilon)^H$ and $\rho(x_n) \leq (1-\varepsilon)^H$. Therefore $\lim_{n\to\infty} \rho(x_n) = 1$.

(ii) Suppose that $||x_n||_L \nrightarrow 0$. Then there is an $\varepsilon \in (0,1)$ and a subsequence (x_{n_k}) of (x_n) such that $||x_{n_k}||_L > \varepsilon$ for all $k \in \mathbb{N}$. By the Proposition 4.3 (i), we obtain that $\rho(x_{n_k}) > \varepsilon^H$ for all $k \in \mathbb{N}$. This implies that $\rho(x_{n_k}) \nrightarrow 0$ as $n \to \infty$. Hence $\rho(x_n) \nrightarrow 0$. □

Theorem 4.5. *The space $V_\rho(\lambda;p)$ is a Banach space with respect to Luxemburg norm defined by*

$$||x||_L = \inf\left\{\lambda > 0 : \rho(\frac{x}{\lambda}) \leq 1\right\}.$$

Proof. We show that every Cauchy sequence in $V_\rho(\lambda;p)$ is convergent according to the Luxemburg norm.

Let $(x^n(j))$ be any Cauchy sequence in $V_\rho(\lambda;p)$ and $\varepsilon \in (0,1)$. Thus, there exists n_0 such that $||x_n - x_m||_L < \varepsilon^M$ for all $m,n \geq n_0$. By the Proposition 3.2 (i), we obtain

$$\rho(x^n - x^m) < ||x^n - x^m||_L < \varepsilon^M, \tag{4.1}$$

for all $n,m \geq n_0$, that is;

$$\sum_{k=1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{j \in I_k} |x^n(j) - x^m(j)| \right)^{p_k} < \varepsilon$$

for all $m,n \geq n_0$. For fixed $j \in \mathbb{N}$, the last inequality gives that

$$|x^n(j) - x^m(j)| < \varepsilon$$

for all $m,n \geq n_0$. Hence we obtain that the sequence $(x^n(j))$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $x^m(j) \to x(j)$ as $m \to \infty$. Therefore, we have

$$\sum_{k=1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{j \in I_k} |x^n(j) - x(j)| \right)^{p_k} < \varepsilon$$

for all $n \geq n_0$.

It remains to show that the sequence $(x(j))$ is an element of $V_\rho(\lambda; p)$. From the inequality (4.1), we can write

$$\sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x^n(j)-x^m(j)|\right)^{p_k}<\varepsilon$$

for all $m, n \geq n_0$. For every $j \in \mathbb{N}$, we have $x^m(j) \to x(j)$, so we obtain that

$$\rho(x^n - x^m) \to \rho(x^n - x)$$

as $m \to \infty$. Since for all $n \geq n_0$,

$$\sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x^n(j)-x^m(j)|\right)^{p_k}\to\sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x^n(j)-x(j)|\right)^{p_k}$$

as $m \to \infty$, then by (4.1) we have $\rho(x^n - x) < \|x^n - x\|_L < \varepsilon$ for all $n \geq n_0$. This means that $x_n \to x$ as $n \to \infty$. So, we have $(x_{n_0} - x) \in V_\rho(\lambda; p)$. Since $V_\rho(\lambda; p)$ is a linear space, we have $x = x_{n_0} - (x_{n_0} - x) \in V_\rho(\lambda; p)$. Therefore the sequence space $V_\rho(\lambda; p)$ is a Banach space with respect to Luxemburg norm. This completes the proof. □

Next, we will show that the space $V_\rho(\lambda; p)$ has Kadec-Klee property. To do this, we need the following Proposition.

Proposition 4.6. *Let $x \in V_\rho(\lambda; p)$ and $(x_n) \subseteq V_\rho(\lambda; p)$. If $\rho(x_n) \to \rho(x)$ as $n \to \infty$ and $x_n(j) \to x(j)$ as $n \to \infty$ for all $j \in \mathbb{N}$, then $x_n \to x$ as $n \to \infty$.*

Proof. Let $\varepsilon > 0$. Since $\rho(x) = \sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x(i)|\right)^{p_k} < \infty$, there exists $j \in \mathbb{N}$ such that

$$\sum_{k=n_0+1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x(j)|\right)^{p_k}<\frac{\varepsilon}{6K} \tag{4.2}$$

where $K = \max\{1, 2^{H-1}\}$.

Since $\rho(x_n) - \sum_{k=1}^{n_0}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x_n(j)|\right)^{p_k} \to \rho(x) - \sum_{k=1}^{n_0}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x(j)|\right)^{p_k}$ as $n \to \infty$ and $x_n(j) \to x(j)$ as $n \to \infty$ for all $j \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that

$$\left|\sum_{k=n_0+1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x_n(j)|\right)^{p_k}-\sum_{k=n_0+1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x(j)|\right)^{p_k}\right|<\frac{\varepsilon}{3K} \tag{4.3}$$

for all $n \geq n_0$. Also, since $x_n(j) \to x(j)$ for all $j \in \mathbb{N}$, we have $\rho(x_n) \to \rho(x)$ as $n \to \infty$. Hence for all $n \geq n_0$, we have $|x_n(j) - x(j)| < \varepsilon$. As a result, for all $n \geq n_0$,

we have

$$\sum_{k=1}^{n_0}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x_n(j)-x(j)|\right)^{p_k}<\frac{\varepsilon}{3}. \tag{4.4}$$

Then from (4.2), (4.3) and (4.4) it follows that for $n \geq n_0$,

$$\begin{aligned}
\rho(x_n-x) &= \sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x_n(j)-x(j)|\right)^{p_k}\\
&= \sum_{k=1}^{n_0}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x_n(j)-x(j)|\right)^{p_k}+\sum_{k=n_0+1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x_n(j)-x(j)|\right)^{p_k}\\
&< \frac{\varepsilon}{3}+K\left[\sum_{k=n_0+1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x_n(j)|\right)^{p_k}+\sum_{k=n_0+1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x(j)|\right)^{p_k}\right]\\
&= \frac{\varepsilon}{3}+K\left[\rho(x_n)-\sum_{k=1}^{n_0}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x_n(j)|\right)^{p_k}+\sum_{k=n_0+1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x(j)|\right)^{p_k}\right]\\
&< \frac{\varepsilon}{3}+K\left[\rho(x)-\sum_{k=1}^{n_0}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x_n(j)|\right)^{p_k}+\frac{\varepsilon}{3K}+\sum_{k=n_0+1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x(j)|\right)^{p_k}\right]\\
&= \frac{\varepsilon}{3}+K\left[\sum_{k=n_0+1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x(j)|\right)^{p_k}+\frac{\varepsilon}{3K}+\sum_{k=n_0+1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x(j)|\right)^{p_k}\right]\\
&< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.
\end{aligned}$$

This shows that $\rho(x_n-x)\to 0$ as $n\to\infty$. Hence by Proposition 4.4 (ii), we have $||x_n-x||_L \to 0$ as $n\to\infty$. □

Now, we give one of the main result of this paper involving geometric properties of the space $V_\rho(\lambda;p)$.

Theorem 4.7. *The space $V_\rho(\lambda;p)$ has the Kadec-Klee property.*

Proof. Let $x\in S(V_\rho(\lambda;p))$ and $(x_n)\subseteq B(V_\rho(\lambda;p))$ such that $||x_n||_L\to 1$ and $x_n\xrightarrow{w}x$ as $n\to\infty$. From Proposition 4.2 (ii), we have $\rho(x)=1$, so it follows from Proposition 4.4 (i) that $\rho(x_n)\to\rho(x)$ as $n\to\infty$. Since $x_n\xrightarrow{w}x$ and the i^{th}-coordinate mapping $\pi_j: V_\rho(\lambda;p)\to\mathbb{R}$ defined by $\pi_j(x)=x(j)$ is continuous linear function on $V_\rho(\lambda;p)$, it follows that $x_n(j)\to x(j)$ as $n\to\infty$ for all $j\in\mathbb{N}$. Thus, by Proposition 4.6 that $x_n\to x$ as $n\to\infty$. □

We prove the following theorem regarding the Banach-Saks of type p property.

Theorem 4.8. *The space $V_p(\lambda)$ has the Banach-Saks of type p.*

Proof. From the Theorem 3.1 b), it is known that the space $V_p(\lambda)$ is a Banach space with respect to the norm $||x||_{V_p(\lambda)}$.

Let (ε_n) be a sequence of positive numbers for which $\sum\limits_{n=1}^{\infty} \varepsilon_n \leq \frac{1}{2}$. Let (x_n) be a weakly null sequence in $B(V_p(\lambda))$. Set $b_0 = x_0 = 0$ and $b_1 = x_{n_1} = x_1$. Then there exists $m_1 \in \mathbb{N}$ such that

$$\left\| \sum_{i=m_1+1}^{\infty} b_1(i) e^{(i)} \right\|_{V_p(\lambda)} < \varepsilon_1.$$

Since (x_n) is a weakly null sequence implies $x_n \to 0$ (coordinatewise), there is an $n_2 \in \mathbb{N}$ such that

$$\left\| \sum_{i=0}^{m_1} x_n(i) e^{(i)} \right\|_{V_p(\lambda)} < \varepsilon_1,$$

where $n \geq n_2$. Set $b_2 = x_{n_2}$. Then there exists an $m_2 > m_1$ such that

$$\left\| \sum_{i=m_2+1}^{\infty} b_2(i) e^{(i)} \right\|_{V_p(\lambda)} < \varepsilon_2.$$

By using the fact that $x_n \to 0$ (coordinatewise), there exists an $n_3 > n_2$ such that

$$\left\| \sum_{i=0}^{m_2} x_n(i) e^{(i)} \right\|_{V_p(\lambda)} < \varepsilon_2,$$

where $n \geq n_3$.

If we continue this process, we can find two increasing subsequences (m_i) and (n_i) such that

$$\left\| \sum_{i=0}^{m_j} x_n(i) e^{(i)} \right\|_{V_p(\lambda)} < \varepsilon_j,$$

for each $n \geq n_{j+1}$ and

$$\left\| \sum_{i=m_j+1}^{\infty} b_j(i) e^{(i)} \right\|_{V_p(\lambda)} < \varepsilon_j,$$

where $b_j = x_{n_j}$. Hence,

$$\left\| \sum_{j=0}^{n} b_j \right\|_{V_p(\lambda)} = \left\| \sum_{j=0}^{n} \left(\sum_{i=0}^{m_{j-1}} b_j(i) e^{(i)} + \sum_{i=m_{j-1}+1}^{m_j} b_j(i) e^{(i)} + \sum_{i=m_j+1}^{\infty} b_j(i) e^{(i)} \right) \right\|_{V_p(\lambda)}$$

$$\leq \left\| \sum_{j=0}^{n} \left(\sum_{i=m_{j-1}+1}^{m_j} b_j(i) e^{(i)} \right) \right\|_{V_p(\lambda)} + \left\| \sum_{j=0}^{n} \left(\sum_{i=0}^{m_{j-1}} b_j(i) e^{(i)} \right) \right\|_{V_p(\lambda)} + \left\| \sum_{j=0}^{n} \left(\sum_{i=m_j+1}^{\infty} b_j(i) e^{(i)} \right) \right\|_{V_p(\lambda)}.$$

$$\leq \left\| \sum_{j=0}^{n} \left(\sum_{i=m_{j-1}+1}^{m_j} b_j(i) e^{(i)} \right) \right\|_{V_p(\lambda)} + 2 \sum_{j=0}^{n} \varepsilon_j.$$

On the other hand since,

$\|x_n\|_{V_p(\lambda)} = \left(\sum_{k=1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{j \in I_k} |x_{n_k}(j)| \right)^p \right)^{\frac{1}{p}}$, it can be seen that $\|x_n\|_{V_p(\lambda)} < 1$. Therefore $\|x_n\|_{V_p(\lambda)}^p < 1$. We have

$$\begin{aligned}
\left\| \sum_{j=0}^{n} \left(\sum_{i=m_{j-1}+1}^{m_j} b_j(i) e^{(i)} \right) \right\|_{V_p(\lambda)}^p &= \sum_{j=0}^{n} \sum_{i=m_{j-1}+1}^{m_j} \left(\frac{1}{\lambda_i} \sum_{v \in I_i} |b_j(v)| \right)^p \\
&\leq \sum_{j=0}^{n} \sum_{i=0}^{\infty} \left(\frac{1}{\lambda_i} \sum_{v \in I_i} |b_j(v)| \right)^p \\
&\leq (n+1).
\end{aligned}$$

Hence we obtain,

$$\left\| \sum_{j=0}^{n} \left(\sum_{i=m_{j-1}+1}^{m_j} b_j(i) e^{(i)} \right) \right\|_{V_p(\lambda)} \leq (n+1)^{\frac{1}{p}}.$$

By using the fact $1 \leq (n+1)^{\frac{1}{p}}$ for all $n \in \mathbb{N}$ and $1 \leq p < \infty$, we have

$$\left\| \sum_{j=0}^{n} b_j \right\|_{V_p(\lambda)} \leq (n+1)^{\frac{1}{p}} + 1 \leq 2(n+1)^{\frac{1}{p}}.$$

Hence $V_p(\lambda)$ has the Banach-Saks type p. This completes the proof of the theorem. □

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